Analysis of Asymmetric Composite Laminates

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Linear laminated plate theory is shown to be inadequate for analysis of asymmetric composite laminates, even in the small deflection range. The von Kármán plate theory was used to analyze composite laminates under in-plane and transverse loadings. For cylindrical bending problems, the governing equations were reduced to linear differential equations with nonlinear boundary conditions yielding a simple solution procedure. Cross-plied laminates were used as examples.

Nomenclature

A_{11} = in-plane stiffness of laminate
B_{11} = extension-bending coupling coefficient
D_{11} = bending stiffness of laminate
M_x = bending moment per unit length
N_x, N_x^o = in-plane force per unit length
a = half-span of the laminate
h = laminate thickness
k = $N_x^o/(D_{11} - B_{11}^2/A_{11})$
q = transverse load
q_o = $q/(D_{11} - B_{11}^2/A_{11})$
u = in-plane displacement at midplane of the laminate
w = transverse deflection
x,y = coordinates
ε_x = in-plane strain

I. Introduction

ASYMMETRIC composite laminates are seldom used in structures, mainly because of the difficulty in controlling their configuration after curing. In reality, however, asymmetric laminates may result from delamination and surface damage in symmetric laminates. This necessitates analysis of asymmetric laminates. Within linear classical lamination theory, the property of asymmetric laminates has been discussed by a number of investigators.[1-4] Their main concern was that bending and middle-plane stretching are coupled when the plate is composed of layers stacked asymmetrically about the middle surface. A recent paper by Yin[5] used large deflection theory in dealing with cylindrical buckling of asymmetrical laminates in delaminated composite plates.

For symmetric laminates, linear laminated plate theory is quite adequate if transverse deflection is small compared with plate thickness. When large deflection occurs, then transverse deflection induces significant in-plane stresses and bending-extension coupling. However, for asymmetric laminates, bending-extension coupling always exists even for small deflections. This early bending-extension coupling causes linear lamination theory to yield large errors in analyzing asymmetric laminates.[6]

In this study, a simple solution was introduced to solve the problem using large deflection theory to correct the deficiency of the linear theory. It was shown that for cylindrical bending problems, the governing equations for large deflection theory could be reduced to linear equations with nonlinear boundary conditions. As a result, solutions to the large deflection problems could be obtained in a rather simple manner. For comparison purposes, a finite-element plate program based upon Mindlin plate theory and von Kármán's large deflection theory were also used to analyze asymmetric laminates.

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II. Governing Equations for Cylindrical Bending of Plates

Consider an aysmmetric cross-plied laminate subjected to a uniform in-plane load or a uniform transverse load q. For a cylindrical bending-type problem we assume that the governing equations are independent of the y-axis. The equilibrium equations based on the von Kármán large deflection theory are given by Ref. 7:

$$N_{x,x} = 0 \tag{1}$$

$$M_{x,xx} + N_x w_{,xx} + q = 0 \tag{2}$$

From Eq. (1) we conclude that

$$N_x = \text{const} = N_x^o \tag{3}$$

Consequently, Eq. (2) becomes

$$M_{x,xx} + N_x^o w_{,xx} + q = 0 \tag{4}$$

In large deflection plate theory, we have

$$N_x = A_{11}\left(u_{,x} + \frac{1}{2}w_{,x}^2\right) - B_{11}w_{,xx} \tag{5}$$

$$M_x = B_{11}\left(u_{,x} + \frac{1}{2}w_{,x}^2\right) - D_{11}w_{,xx} \tag{6}$$

These definitions can be found in Ref. 8.

Substitution of Eq. (5) into Eq. (6) yields

$$M_x = \frac{B_{11}}{A_{11}}N_x^o + \left(\frac{B_{11}^2}{A_{11}} - D_{11}\right)w_{,xx} \tag{7}$$

Substituting Eq. (7) into Eq. (4), we obtain

$$w_{,xxxx} - k^2 w_{,xx} = q_o \tag{8}$$

where

$$k^2 = N_x^o/(D_{11} - B_{11}^2/A_{11}) \tag{9}$$

$$q_o = q/(D_{11} - B_{11}^2/A_{11}) \tag{10}$$

As is clear from our derivation, u and w are the displacements measured from the original stress-free state. Also note that if Eqs. (5) and (6) are substituted into Eq. (2), then a nonlinear equation results. Dropping the nonlinear terms at this stage would lead to the classical laminated plate theory.

III. Asymmetric Laminates Subjected to In-plane Loading

Consider an asymmetric cross-plied laminate subjected to a uniform in-plane load N_x^o along the two simply supported straight edges $x = \pm a$. The general solution for Eq. (8) with $q = 0$ is

$$w(x) = C_1 \cosh kx + C_2 \sinh kx + C_3 x + C_4 \tag{11}$$

If the origin of the coordinate axis is chosen to locate at the midspan, then $C_2 = C_3 = 0$ because of symmetry. The two constants C_1 and C_4 are determined by the simply supported boundary conditions:

$$w = 0, \qquad M_x = 0 \quad \text{at} \quad x = \pm a \tag{12}$$

from which we obtain

$$C_1 = \frac{B_{11}}{A_{11} \cosh ka}, \qquad C_4 = -\frac{B_{11}}{A_{11}} \tag{13}$$

The in-plane displacement can be obtained by integrating Eq. (5) with $N_x = N_x^o$. The result is

$$u(x) = \frac{B_{11}^2 k}{A_{11} \cosh ka} \sinh kx + \frac{N_x^o}{A_{11}} x \tag{14}$$

The maximum deflection occurs at the midspan ($x = 0$):

$$w_{\max} = \frac{B_{11}}{A_{11}} \left(\frac{1}{\cosh ka} - 1 \right) \tag{15}$$

It is interesting to note that

$$w_{\max} \to -\frac{B_{11}}{A_{11}} \quad \text{as} \quad N_x^o \to \infty \tag{16}$$

that is, there exists a bound for deflection as N_x^o increases.

The longitudinal strain at the midplane is given by

$$\varepsilon_x^o = \frac{B_{11}^2 k^2}{A_{11}^2 \cosh ka} \cosh kx + \frac{N_x^o}{A_{11}} \tag{17}$$

It is noted that the longitudinal strain is not uniform throughout the span. From Eq. (17) we note that for $x \neq \pm a$

$$\varepsilon_x^o \to \frac{N_x^o}{A_{11}} \quad \text{as} \quad N_x^o \to \infty \tag{18}$$

Thus, the bending-extension coupling effect diminishes as N_x^o increases. Hence, for large in-plane loads, strains can be obtained from the governing equations by setting $B_{11} = 0$.

The difference between the above formulation and classical linear laminated plate theory lies in the extra second derivative term in Eq. (8).

For the in-plane loading problem considered above, the solutions according to classical linear theory can be easily obtained. The results are

$$w(x) = \frac{1}{2} \frac{B_{11} N_x^o}{D_{11} A_{11} - B_{11}^2} (x^2 - a^2) \tag{19}$$

$$u(x) = \frac{D_{11} N_x^o}{A_{11} D_{11} - B_{11}^2} x \tag{20}$$

The maximum transverse deflection is given by

$$w_{\max} = \frac{1}{2} \frac{B_{11} N_x^o a^2}{B_{11}^2 - A_{11} D_{11}} \tag{21}$$

Thus, $w_{\max} \to \infty$ as $N_x^o \to \infty$. This is quite different from predictions by the present large deflection theory.

The longitudinal strain at the midplane of the laminate obtained from Eq. (20) is constant over the span, i.e.,

$$\varepsilon_x^o = \frac{D_{11} N_x^o}{A_{11} D_{11} - B_{11}^2} \tag{22}$$

which is quite different from that given by Eq. (17). In particular, for large values of N_x^o, the strain given by Eq. (22) is still affected by the coupling coefficient B_{11} and is larger than that given by Eq. (18).

In the numerical analysis, the elastic constants for a graphite/epoxy composite are given by $E_1 = 20$ msi, $E_2 = 1.4$ msi, $\nu_{12} = 0.30$, $G_{12} = G_{13} = G_{23} = 0.7$ msi. The ply thickness is 0.005 in. The corresponding plate stiffnesses for the $[90_4/0_4]$ laminate are obtained as $A_{11} = 431{,}400$ lb/in., $B_{11} = -3737$ lb-in./in., $D_{11} = 57.53$ lb-in.

In this study, a nine-node isoparametric finite element, based on the Mindlin plate theory and von Kármán large deflection assumptions,[9] is also employed for large deflection analysis. The $[90_4/0_4]$ laminated plate examined in this study measures 9 in. × 1.5 in.

This plate is subjected to a uniform in-plane load N_x^o at the two edges. The edges along the y direction are assumed to be simply supported but movable in the x direction. Twenty-four elements are used for each quadrant. This mesh is shown to yield a converged solution. The Newton-Raphson method is adopted in the numerical analysis, and the iteration procedure is carried out until a convergence tolerance of 0.1% is met.

Figures 1 and 2 present transverse deflection curves for the $[90_4/0_4]$ laminate subjected to different in-plane loads. The finite-element solution agrees very well with the present large deflection solution. It is evident that linear theory yields large errors even if the deflection is small compared with the laminate thickness.

IV. Asymmetric Laminates Subjected to Transverse Loading

Consider an asymmetric cross-plied laminate under uniform transverse load q. For a cylindrical bending type of problem, we assume that the governing equations are independent of the y axis. The equilibrium equations based on the von Kármán large deflection assumption are given by Eq. (8). This equation is a linear equation and can be solved easily. If the boundary conditions are symmetric, then the general solution is

$$w(x) = C_1 \cosh kx + C_2 - \frac{q_o}{2k^2} x^2 \tag{23}$$

For pinned-pinned boundary conditions, we have

$$w(\pm a) = 0$$
$$M_x(\pm a) = 0 \tag{24}$$
$$u(\pm a) = 0$$

Since N_x is an unknown constant along the x axis, the in-plane displacement u can be derived by integrating Eq. (5) over the span of laminates using the general solution Eq. (23). The boundary conditions can be expressed as

$$w(a) = 0 \tag{25}$$

$$M_x(a) = \frac{B_{11}}{A_{11}} N_x^o - \left(D_{11} - \frac{B_{11}^2}{A_{11}}\right) w,_{xx} = 0 \tag{26}$$

$$u(a) = \int_o^a \left(\frac{N_x^o}{A_{11}} + \frac{B_{11}}{A_{11}} w,_{xx} - \frac{1}{2} w,_x^2\right) \mathrm{d}x = 0 \tag{27}$$

Substituting Eq. (23) into Eqs. (25–27) and integrating, we have

$$C_1 = \left(\frac{B_{11}}{A_{11}} + \frac{q_o}{k^4}\right) / \cosh ka \tag{28}$$

$$C_2 = \frac{q_o a^2}{2k} - C_1 \cosh ka \tag{29}$$

$$u(a) = \frac{N_x^o}{A_{11}} a + \frac{B_{11}}{A_{11}} k C_1 \sinh ka - \frac{B_{11}}{A_{11}} \frac{q_o}{k^2} a - \frac{1}{2} k^2 C_1^2 \left[\frac{1}{4k} \sinh 2ka - \frac{a}{2}\right] - \frac{q_o^2 a^3}{6k^4} + C_1 \frac{q_o}{k^3} [ka \cosh ka - \sinh ka] = 0 \tag{30}$$

These three equations contain three unknown quantities: C_1,C_2, and N_x^o. Since Eq. (30) is a transcendental equation, the solution must be solved numerically. The bisection method is used.

The displacement solutions according to the classical linear theory can be obtained from Eq. (8) (by setting $k = 0$) with the proper boundary conditions. The results are

$$w(x) = \frac{q_o}{24}(x^4 - a^4) - \frac{3D_{11}A_{11} - 2B_{11}^2}{12 D_{11} A_{11}} q_o a^2 (x^2 - a^2) \tag{31}$$

$$u(x) = \frac{B_{11}}{6A_{11}} q_o x^3 - \frac{B_{11}}{A_{11}} q_o a^2 x \tag{32}$$

The nonlinear plate finite-element program is also used to solve the problem of transverse loading. A pinned-pinned $[90_4/0_4]$ laminate 9 in. × 1.5 in. is used for analysis. The solution for deflection is taken along the centerline.

Numerical results for the graphite/epoxy $[90_4/0_4]$ laminate are presented in Figs. 3–7. In all cases the present solutions agree with the nonlinear finite-element solutions.

Figure 3 shows the induced in-plane force N_x as a function of the transverse load q. The relation appears to be nonlinear initially.

Figures 4 and 5 present the transverse deflection and in-plane displacement curves for $q = 0.1$ lb/in. obtained using lin-

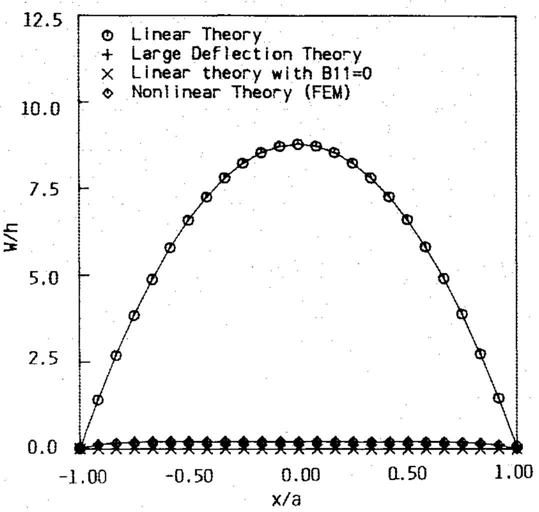

Fig. 1 Out-of-plane deflection of $[90_4/0_4]$ laminate subjected to uniform in-plane load $N_x^o = 100$ lb/in.

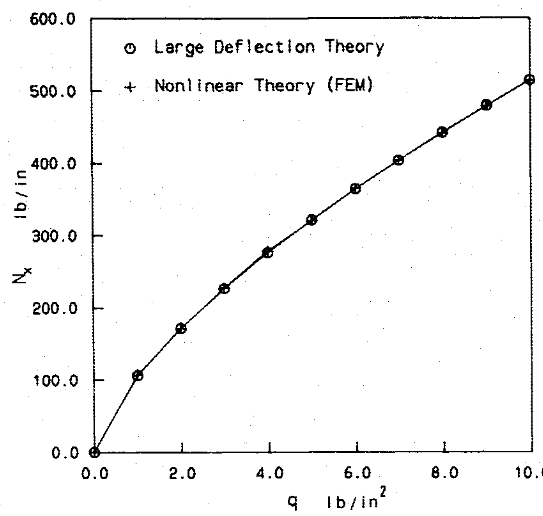

Fig. 3 Membrane resultant N_x of $[90_4/0_4]$ laminate with hinged supports subjected to uniform transverse loads.

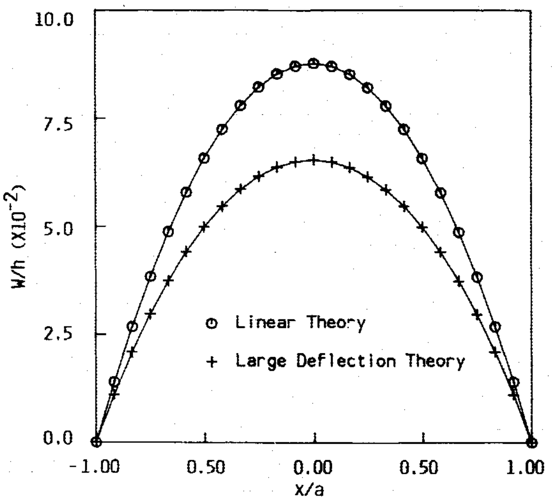

Fig. 2 Out-of-plane deflection of $[90_4/0_4]$ laminate subjected to uniform in-plane load $N_x^o = 1$ lb/in.

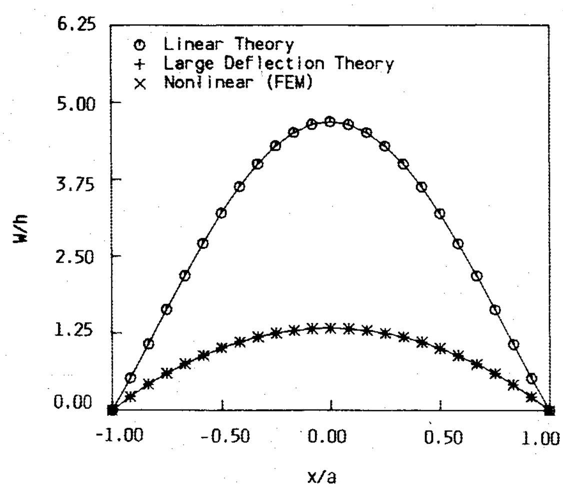

Fig. 4 Out-of-plane deflection of $[90_4/0_4]$ laminate subjected to uniform transverse load $q = 0.1$ lb/in^2.

ear and large deflection theories. Note that the applied load is considered positive if it acts upward. Because of the asymmetric nature of the laminate, positive and negative loads of the same magnitude produce different magnitudes of deflection, as shown in Figs. 6 and 7. Note that, according to the linear theory, these two deflections are of the same magnitude.

Due to extension-bending coupling of the laminate, in-plane force N_x is induced by the transverse load. Figure 8 shows the in-plane forces associated with positive and negative loads. According to the linear theory, a positive load applied on the $[90_4/0_4]$ laminate produces a compressive in-plane force and a negative q produces a tensile N_x. Large deflection theory predicts quite different results. From Fig. 8 it is seen that a positive q produces a negative N_x initially (due to extension-bending coupling), but then becomes positive as q increases due to large deflection. This explains why for small values of positive q, greater deflection is obtained from large deflection theory than from linear theory since a compressive in-plane force has an apparent softening effect (see Fig. 6).

It is also noted that for larger values of q greater in-plane force N_x is obtained from linear theory than from large deflection theory. In large deflection theory, in-plane tensile force provides additional bending rigidity that reduces transverse deflection and, thus, also reduces the in-plane displacement through extension-bending coupling.

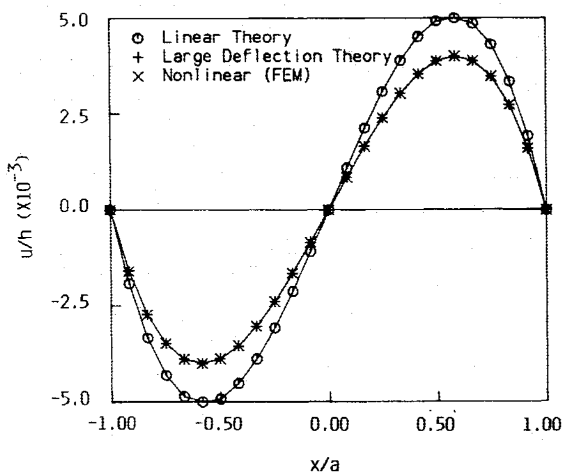

Fig. 5 Axial displacement u_x of $[90_4/0_4]$ laminate with hinged supports subjected to transverse load $q = -0.1$ lb/in^2.

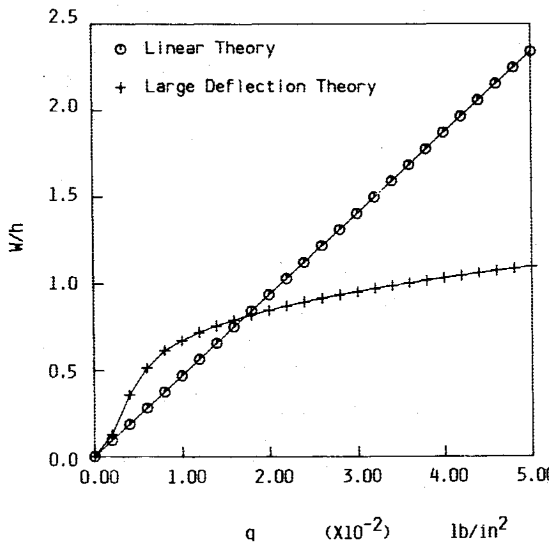

Fig. 6 Maximum deflection of $[90_4/0_4]$ laminate with hinged supports vs positive transverse loads.

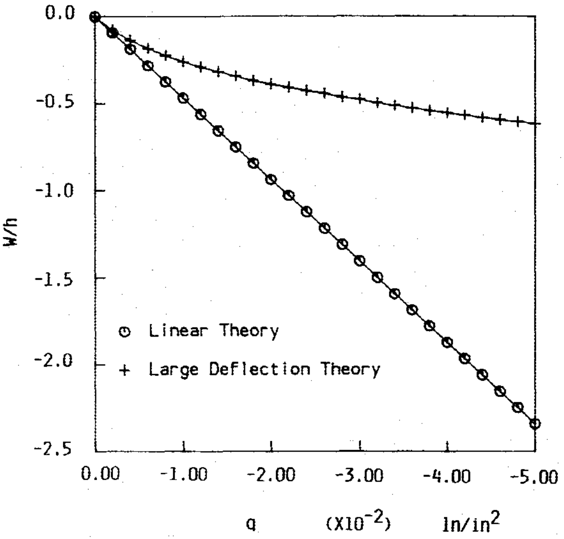

Fig. 7 Maximum deflection of $[90_4/0_4]$ laminate with hinged supports vs negative transverse loads.

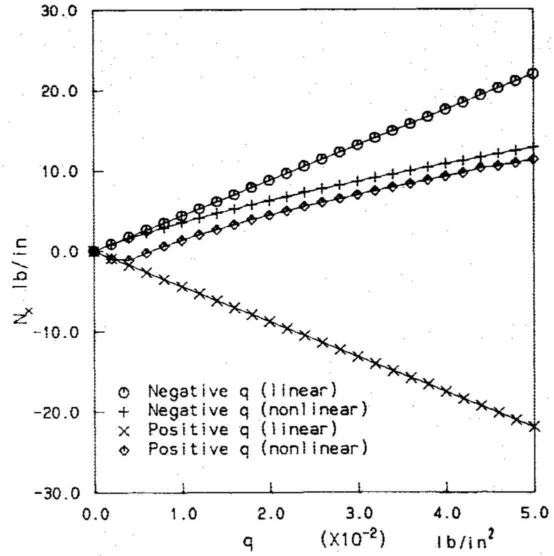

Fig. 8 Membrane force N_x in $[90_4/0_4]$ laminate with hinged supports subjected to uniform transverse loads.

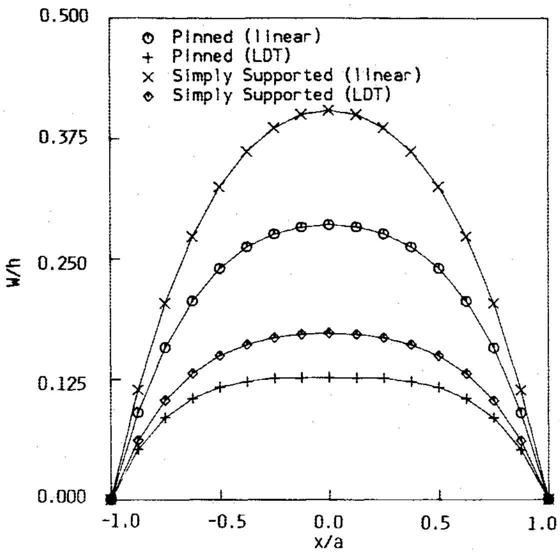

Fig. 9 Out-of-plane deflection along the centerline of a 9 in. × 9 in. $[90_4/0_4]$ laminated plate subjected to uniform in-plane load $N_x^o = 10$ lb/in.

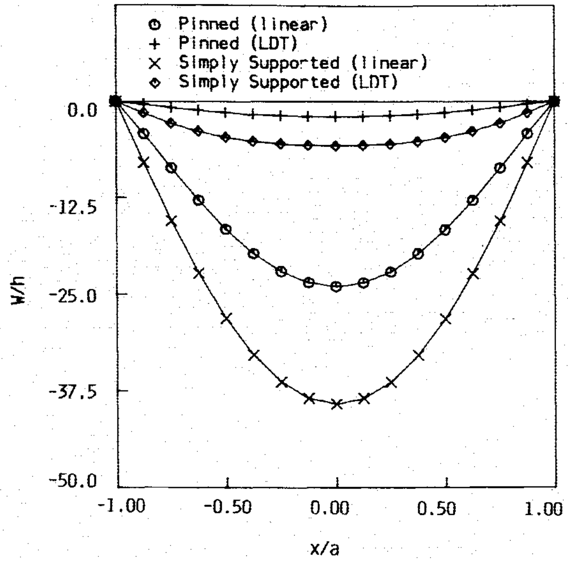

Fig. 10 Out-of-plane deflection along the centerline of a 9 in. × 9 in. $[90_4/0_4]$ laminated plate subjected to uniform transverse load $q = -1$ lb/in^2.

V. Asymmetric Laminates

For plates, the equilibrium equations for in-plane resultant forces are given by

$$N_{x,x} + N_{xy,y} = 0 \tag{33}$$

$$N_{xy,x} + N_{y,y} = 0 \tag{34}$$

where N_x, N_y, and N_{xy} are the in-plane resultant forces. The in-plane forces are not constant throughout the plate. Consequently, the plate governing equations according to large deflection theory cannot be simplified as was previously done for cylindrical bending problems. Thus, the nine-node plate finite-element program is employed to perform the analysis.

A 9 in. × 9 in. $[90_4/0_4]$ graphite/epoxy laminate with various boundary and loading conditions is anlayzed. Because of symmetry, only a quadrant of the plate is used for analysis with 16 elements.

Figure 9 presents the tranverse deflection along the centerline in x direction of the plate subjected to a uniform in-plane load $N_x^0 = 10$ lb/in. The loading edges are simply supported (movable in x direction), and the other two edges are either simply supported or pinned. Again, the linear and nonlinear solutions show great discrepancies.

Figure 10 shows the deflection profiles of the laminate for two different boundary conditions, i.e., simply supported and pinned along the four edges, respectively. The loading is a uniform transverse load $q = 1$ lb/in^2.

VI. Conclusion

From this study it is concluded that in analyzing asymmetric composite laminates with strong extension-bending coupling, large deflection plate theory must be used even for deflections that are normally considered small. For cylindrical bending problems, it is found that the plate displacement equilibrium equations according to large deflection theory can be expressed as linear equations for the deflection, leaving nonlinear boundary conditions. This linearity of the differential equations greatly simplifies the large deflection analysis.

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